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A SHORT PROOF OF A CHERNOFF INEQUALITY

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Abstract. Chernoff proves an inequality using Hermite polynomials. Here we prove and generalize this inequality using Cauchy-Schwartz inequality and Fubini equality.

Let X be a Gaussian random variable $N(0, 1)$ and f an absolutel continuous real function of real variable, with derivative f' such that

(a) $\mathrm{E}(f^2(X)) < \infty$, (b) $E(f'^2(X)) < \infty$,

THEOREM 1. *The function f, described above, verifies*

$$\operatorname{Var} f(X) \leqslant \mathrm{E}(f'^2(X)). \tag{1}$$

The equality in (1) *occurs if and only if f is an affine function, i.e. if ther exist two real numbers a and b such that $f(X) = ax+b$.*

Inequality (1) was established by Chernoff[1] and proved by him with the use of Hermite polynomials. In the sequel we apply Cauchy-Schwartz inequality and Fubini equality to give a refinement and a generalization of Chernoff's inequality.

LEMMA. *For any absolutely continuous real function verifying assumption* (a) *and* (b) *we have*

$$\mathrm{E}(f(X)-f(0))^2 \leqslant \mathrm{E}(f'^2(X)). \tag{2}$$

The equality in (2) *occurs if and only if there exist three real numbers a_1, a_2, and b such that*

$$f(X) = a_1 x \mathbf{1}_{]-\infty,0[}(x) + a_2 x \mathbf{1}_{]0,\infty[}(x) + b.$$

Theorem 1 is a straight forward consequence of this lemma.

Proof of the lemma. Let us denote by g the probability density of X.

[1] H. Chernoff, *A note on an inequality involving the normal distribution*, Ann. Prob. 9 (1981), p. 533-535.

In order to prove (2) we observe that

$$\mathrm{E}\big(f(X)-f(0)\big)^2 = \int_{\boldsymbol{R}} \Big(\int_0^x f'(u)\,du\Big)^2 g(x)\,dx$$

$$= \int_0^\infty \Big(\int_0^\infty \mathbf{1}_{]0,x[}(u) f'(u)\,du\Big)^2 g(x)\,dx + \int_{-\infty}^0 \Big(-\int_{-\infty}^0 \mathbf{1}_{]x,0[}(u) f'(u)\,du\Big)^2 g(x)\,dx.$$

The interval $]0, x[$ (or $]x, 0[$) is a finite positive measure space for Lebesgue measure du. Hence, using Cauchy-Schwarz inequality, we obtain

$$\mathrm{E}\big(f(X)-f(0)\big)^2 \leqslant \int_0^\infty \Big(\int_0^\infty x\mathbf{1}_{]0,x[}(u) f'^2(u)\,du\Big) g(x)\,dx +$$

$$+ \int_{-\infty}^0 \Big(\int_{-\infty}^0 -x\mathbf{1}_{]x,0[}(u) f'^2(u)\,du\Big) g(x)\,dx.$$

Observe that

$$\mathbf{1}_{]0,x[}(u) = \mathbf{1}_{]u,\infty[}(x) \quad \text{if } x > 0,$$

$$\mathbf{1}_{]x,0[}(u) = \mathbf{1}_{]-\infty,u[}(x) \quad \text{if } x < 0,$$

and

$$\int_0^\infty x\mathbf{1}_{]u,\infty[}(x) g(x)\,dx = g(u), \tag{3}$$

$$\int_{-\infty}^0 -x\mathbf{1}_{]-\infty,u[}(x) g(x)\,dx = g(u). \tag{4}$$

Using the Fubini equality, we easily get (2).

In order to prove the second part of Lemma by using Cauchy-Schwartz inequality, it is sufficient to observe that inequality (2) becomes an equality if and only if

$$\mathbf{1}_{]0,x[}(u) f'(u) = a_2\,\mathbf{1}_{]0,x[}(u) \quad \text{for any } x \in \boldsymbol{R}_+,$$

$$\mathbf{1}_{]x,0[}(u) f'(u) = a_1\,\mathbf{1}_{]x,0[}(u) \quad \text{for any } x \in \boldsymbol{R}_-,$$

q.e.d.

GENERALISATION

THEOREM 2. (i) *Let X be a random variable with probability density*

$$g(x) = \lambda_1\,\mathbf{1}_{]-\infty,0[}(x) \exp\Big\{-\frac{|x|^p}{p}\Big\} + \tag{5}$$

$$+\lambda_2\,\mathbf{1}_{]0,\infty[}(x) \exp\Big\{-\frac{|x|^p}{p}\Big\}, \quad \textit{where } p \geqslant 1,$$

and f an absolutely continuous real function of a real variable, with derivative f', verifying

$$\text{(a}'\text{)}\ \mathrm{E}\left(|f(X)|^p\right) < \infty, \qquad \text{(b}'\text{)}\ \mathrm{E}\left(|f'(X)|^p\right) < \infty.$$

Then the inequality

$$\mathrm{E}\,|f(X) - f(0)|^p \leqslant \mathrm{E}\left(|f'(X)|^p\right) \tag{6}$$

holds, and the equality occurs if and only if there exist a_1, a_2, and b such that

$$f(x) = a_1\, x \mathbf{1}_{]-\infty,0[}(x) + a_2\, x \mathbf{1}_{]0,\infty[}(x) + b.$$

(ii) *If X is a real random variable such that* (6) *holds for any absolutely continuous real function of a real variable, verifying* (a′) *and* (b′), *then the probability density of X is defined by* (5).

The proof is similar to that of the lemma.

Giving to g the form (1), and using Hölder's inequality instead of Cauchy-Schwartz inequality, equalities (3) and (4) become

$$\int_0^\infty |x|^{p-1}\, \mathbf{1}_{]u,+\infty[}(x)\, g(x)\, dx = g(u), \qquad u > 0, \tag{7}$$

$$\int_0^\infty -|x|^{p-1}\, \mathbf{1}_{]-\infty,u[}(x)\, g(x)\, dx = g(u), \qquad u < 0. \tag{8}$$

Then (i) is proved.

To prove (ii) we observe that (6) occurs if and only if equalities (7) and (8) hold. Then g is absolutely continuous and verify the differential equation

$$\frac{g'(u)}{g(u)} = -|u|^{p-1},$$

which implies (ii).

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